

# CHARACTERISTIC CLASSES IN $TMF$ OF LEVEL 3

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ABSTRACT. We compute the  $TMF_1(3)$  cohomology of the 7-connective cover  $BString$  of  $BO$ . It turns out that it is freely generated by Pontryagin classes and another class when appropriately completed. The Pontryagin classes come from  $BSpin$  and freely generate the  $TMF_1(3)$  characteristic classes for  $Spin$  bundles. As a first application we show how to construct  $TMF_1(n)$  cohomology classes from stable positive energy representations of the loop groups  $LSpin$ .

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Characteristic classes are cohomology classes which are naturally associated to principal  $G$  bundles on topological spaces. Examples include Stiefel-Whitney classes for  $O(n)$ -bundles or  $KO$ -Pontryagin classes for  $SO(n)$  bundles. If the cohomology theory is oriented with respect to  $G$ -bordism the characteristic classes can be evaluated on fundamental classes of manifolds with  $G$ -structure and yield characteristic numbers. It is well known that oriented bordism is determined by Stiefel-Whitney numbers and  $H\mathbb{Z}$ -Pontryagin numbers whereas  $Spin$  bordism is determined by those numbers and  $KO$ -Pontryagin numbers.

In this paper we consider characteristic classes for  $Spin$  and  $String$  bundles. Here,  $String$  is the 6-connected cover of the  $Spin$  group. The cohomology theory is  $TMF_1(3)$ , the cohomology of topological modular forms of level 3. There is the general hope that the  $String$  bordism can be determined by Stiefel-Whitney,  $H\mathbb{Z}$ - and  $TMF$ - characteristic numbers. At the prime 2 the connective version of  $TMF$  should split off the Thom spectrum  $MString$  and there is some evidence that another summand is given by the 16th suspension of the connective cover of  $TMF_0(3)$  (compare [MR09]). However, in order to get a map a better understanding of the characteristic classes is necessary. (The relation between  $TMF_1(3)$  and  $TMF_0(3)$  is described in section 2.)

The homology groups  $E_*BU\langle 6 \rangle$  for complex oriented theories  $E$  have been described in [AHS01] in terms cubical structures on the formal group of  $E$ . Here,  $BU\langle 6 \rangle$  is the complex analogue of  $BString$ . It has been conjectured that the real case  $E_*BString$  admits a similar description in terms of “real” cubical structures.

The cohomology groups  $E^*BString$  for various complex oriented versions of  $TMF$  have been considered in [Lau99]. When 2 is invertible these groups have been computed by the topological  $q$ -expansion principle. The difficulty appears at the prime 2 since the  $String$  bordism groups have 2-torsion which has not been computed yet. This paper deals with this remaining prime.

We first look at characteristic classes for  $Spin$  bundles and show the following  $q$ -expansion principle:

**Theorem 1.1.** *The diagram*

$$\begin{array}{ccc} TMF_1(3)^* BSpin & \xrightarrow{\lambda} & K_{Tate}^* BSpin \\ \downarrow & & \downarrow ch \\ H^*(BSpin, TMF_1(3)_{\mathbf{Q}}^*) & \longrightarrow & H^*(BSpin, K_{Tate}^*_{\mathbf{Q}}) \end{array}$$

*is a pullback*

The horizontal map is given by the Miller character which corresponds to the evaluation on the Tate curve on the moduli stack of elliptic curves. On coefficients this map is just the traditional  $q$ -expansion map for modular forms. The theory  $K_{Tate}$  is the power series ring  $K[1/3]((q))$  of  $K$ -theories. The right vertical map is the Chern character and the left vertical map is the Dold character, that is, the rationalization.

The theorem determines the ring of  $TMF_1(3)$  characteristic classes for  $Spin$  bundles. It says that each natural power series of stable vector bundles with  $Spin$  structure group which is natural, stable and whose Chern character gives a modular form (that is, it is invariant under the appropriate Möbius transformations) gives rise to a unique class in the  $TMF_1(3)$  cohomology of its base space.

The proof of the theorem is given in section 3. A splitting principle which is based on the Kitchloo-Laures computation [KL02] of the Morava  $K(2)$  homology of  $BSpin$  forces the universal coefficient spectral sequence to collapse. Its  $E_2$  term is studied via a chromatic resolution. The pullback property is reduced via the universal coefficient isomorphisms to the classical  $q$ -expansion principle.

The result allows the construction of Pontryagin classes with their usual properties:

**Theorem 1.2.** (i) *There are unique classes  $p_i \in TMF_1(3)^{4i} BSpin$  with the following property: the formal series  $p_t = 1 + p_1 t + p_2 t^2 \dots$  is given by*

$$\prod_{i=1}^m (1 + t \rho^*(x_i \bar{x}_i))$$

*when restricted to the classifying space of each maximal torus of  $Spin(2m)$ . Here,  $\rho$  is the map from the maximal torus of  $SO(2m)$  and the  $x_i$  (and  $\bar{x}_i$ ) are the first Conner-Floyd Chern classes of the line bundles  $L_i$  (resp.  $\bar{L}_i$ ).*

(ii) *The classes  $p_i$  freely generate the  $TMF_1(3)$  cohomology of  $BSpin$ , that is,*

$$TMF_1(3)^* BSpin \cong TMF_1(3)^* \llbracket p_1, p_2, \dots \rrbracket.$$

Next we turn to the calculation of the characteristic classes for  $BString$ . Here it turns out that the  $q$ -expansion principle only holds for odd primes. A  $String$  characteristic class may not even be determined by its characters. However we have the

**Theorem 1.3.** *Let  $\widehat{TMF}_1(3)$  be the  $K(2)$  localization of  $TMF_1(3)$  at the prime 2. Then there is an isomorphism of algebras*

$$\widehat{TMF}_1(3)^* BString \cong \widehat{TMF}_1(3)^* \llbracket r, p_1, p_2, \dots \rrbracket$$

*where  $p_1, p_2, \dots$  are the Pontryagin classes coming from  $BSpin$  and  $r$  restricts to a topological generator of degree 6 in the  $K(2)$  cohomology of  $K(\mathbb{Z}, 3)$ .*

The theorem relies on a description of the Morava  $K(2)$  homology of  $BString$  given by Kitchloo-Laures-Wilson in [KLW04a][KLW04b]. Its proof will be given in section 4.

As a first consequence we give a proof of the stable version of a conjecture by Brylinski which was stated in [Bry90]. The precise formulation is given in the last section. For big  $n$  divisible by 3 there is a group homomorphism

$$\varphi : (P_m)_{\Gamma(n)} \longrightarrow TMF(n)^* BSpin$$

from a stable group of positive energy representations  $V$  of  $\tilde{LSpin}$  to the  $TMF$  cohomology with level  $n$  structure. In terms of its Miller character the map is given by the bundle obtained by associating the representation to the principal  $\tilde{LSpin}$  bundle over  $BString$ :

$$\lambda\varphi_d(V) = L \widetilde{ESpin} \times_{\tilde{LSpin}} V.$$

Its evaluation on the fundamental class of a *String* manifold is the “index of the Dirac operator on  $LM$  with coefficient in the bundle” associated to the representation  $V$ .

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## 2. TOPOLOGICAL MODULAR FORMS OF LEVEL 3

In this section we describe the spectrum of topological modular forms with level structures. Most of the results are well known and are collected for later reference. We start more general than actually needed in order to put the main theorem in a broader picture.

Let  $\mathcal{M}$  denote the stack of smooth elliptic curves. A morphism  $f : S \longrightarrow \mathcal{M}$  determines an elliptic curve  $C_f$  over the base scheme  $S$  (see Deligne and Rapoport [DR75].) The cotangent spaces of the identity section of each elliptic curve give rise to a line bundle  $\omega$  on  $\mathcal{M}$ . A section of  $\omega^{\otimes k}$  is called a modular form of weight  $k$ . We write  $M_k$  for the ring of modular forms of weight  $k$ .

Let  $C$  be an elliptic curve over a ring  $R$ . Let  $C[n]$  denote the kernel of the self map  $[n]$  which multiplies by  $n$  on  $C$ . If  $n$  is invertible in  $R$  then  $C[n]$  is of the form  $\mathbb{Z}/n \times \mathbb{Z}/n$ . A choice of an isomorphism is called a  $\Gamma(n)$  structure on  $C$ . A monomorphism  $\mathbb{Z}/n \longrightarrow C[n]$  is a  $\Gamma_1(n)$  structure. It corresponds to a choice of a point of exact order  $n$ . A choice of a subgroup scheme  $A$  of  $C[n]$  which is isomorphic to  $\mathbb{Z}/n$  is called a  $\Gamma_0(n)$  structure. There are maps of associated moduli stacks

$$(1) \quad \mathcal{M}_{\Gamma(n)} \longrightarrow \mathcal{M}_{\Gamma_1(n)} \longrightarrow \mathcal{M}_{\Gamma_0(n)} \longrightarrow \mathcal{M}$$

and the sections of the associated line bundles are called modular forms with the corresponding level structures. For example, a  $\Gamma_1(n)$  modular form  $f$  of weight  $k$  associates for each triple  $(C/R, \omega, P)$  where  $C$  is an elliptic curve over  $R = R[1/n]$ ,  $\omega$  is a translation invariant nowhere vanishing differential and  $P$  is a point of order  $n$ , an element in the ring  $R$ . This association should only depend on the isomorphism class of the triple, it should be invariant under base change and it should satisfy

$$f(C, a\omega, P) = a^{-k} f(C, \omega, P)$$

for all  $a \in R^\times$ .

The case  $n = 3$  can be made more explicit. Locally any such triple  $(C, \omega, P)$  can uniquely be written in the form

$$(2) \quad C : y^2 + a_1xy + a_3y = x^3$$

in a way that  $P$  is the origin  $(0, 0)$  and  $\omega$  has the standard form  $\omega = dx/2y + a_1x + a_3$ . The proof of this fact can be found in [MR09]3.2. It means that there is universal triple  $(C, \omega, (0, 0))$  over the ring  $\mathbb{Z}[1/3, a_1, a_3, \Delta^{-1}]$  with the property that locally any other triple is obtained by base change. Hence a  $\Gamma_1(3)$  modular form is determined by this object and gives an element in this ring. Moreover, any element in the ring gives a modular form with level structure. We get

$$(3) \quad M_{\Gamma_1(n)_*} \cong \mathbb{Z}[1/3, a_1, a_3, \Delta^{-1}].$$

One should mention that all moduli problems for  $\Gamma_1(n)$  and  $\Gamma(n)$  structures with  $n \geq 3$  are representable this way but  $\Gamma_0(n)$  is not representable. A good reference for these classical results on modular forms and level structures is the book [KM85].

There is a derived version of these concepts as follows:

**Theorem 2.1** (Goerss, Hopkins, Lurie, Miller et al [Goe10]). *There is a sheaf  $\mathcal{O}_{TMF}$  of  $E_\infty$  ring spectra over  $\mathcal{M}$  in the étale topology. This sheaf satisfies:*

- (i) *The spectrum  $TMF = \Gamma \mathcal{O}_{TMF}$  only has 2 and 3 torsion in homotopy. Away from 6 it is concentrated in even degrees and we have an isomorphism*

$$\pi_{2k} TMF[1/6] \cong M_k[1/6].$$

- (ii) *There is an orientation map  $MString \rightarrow TMF$  which induces the Witten genus in homotopy. In fact, its image coincides with the homotopy groups of a connective version of  $TMF$ . An open question is if this map splits.*
- (iii) *The sequence (1) of moduli stacks gives a sequence of spectra*

$$TMF(n) \longleftarrow TMF_1(n) \longleftarrow TMF_0(n) \longleftarrow TMF$$

*and induced isomorphisms*

$$TMF[1/n] \cong TMF(n)^{hGl_2(\mathbb{Z}/n \times \mathbb{Z}/n)}$$

$$TMF_0(n) \cong TMF_1(n)^{hGl_1(\mathbb{Z}/n)}$$

- (iv) *For all  $f : Spec(R) \rightarrow \mathcal{M}$  the spectrum  $E = \Gamma f^* \mathcal{O}_{TMF}$  is an even periodic ring spectrum whose formal group  $E^0 BS^1$  is equipped with an isomorphism to the formal completion of  $C_f$ .*

The spectrum  $TMF_1(3)$  can be described much more elementary if one is only interested in its cohomology theory. Since the moduli problem is representable its homotopy coincides with the ring  $M_{\Gamma_1(3)}$  by property (iv). This ring has been determined in (3). Explicitly, when we choose a coordinate on the formal group of the curve (2) we obtain a formal group law. These are classified by a map of the Lazard ring  $L$  which coincides with the homotopy groups of complex bordism  $MU$ . For instance, for the standard coordinate we have for the Hazewinkel generators (see [Lau04] Lemma 1) at  $p = 2$

$$(4) \quad v_1 = a_1$$

$$(5) \quad v_2 = a_3.$$

The Hazewinkel generators form a regular sequence and hence satisfy the Landweber exactness conditions. Thus for finite complexes  $X$  we have natural isomorphisms

$$TMF_1(3)^*X \cong MU^*X \otimes_{MU^*} M_{\Gamma_1(3)}.$$

(In the older literature as [Bry90, Fra92, Bak94] this theory carried the names  $Ell^{\Gamma_1(3)}$  or  $E^{\Gamma_1(3)}$ .) Locally at the prime 2, the spectrum  $TMF_1(3)$  is closely related to a generalized Johnson-Wilson theory  $E(2)$  in the sense of [LN12]:

**Lemma 2.2.** *The map*

$$E(2)_*[(v_1^3 - 27v_2)^{-1}] \longrightarrow TMF_1(3)_{(2)*}$$

*is an isomorphism.*

*Proof.* The discriminant of the universal curve has the form

$$\Delta = a_3^3(a_1^3 - 27a_3).$$

The result follows from (4) and (5).  $\square$

**Convention 2.3.** In the sequel we will use the notation  $E(2)$  for any generalized Johnson-Wilson spectrum with a height 2 formal group law. Moreover, it shall be remarked that all of our results for  $TMF_1(3)$  are valid without inverting the class  $(a_1^3 - 27a_3)$ . That is, we may allow certain singularities for the elliptic curves without changing the notation of the resulting theory  $TMF_1(3)$ .

**Remark 2.4.** In [LN12] Naumann and Lawson construct a connective version of  $TMF_1(3)$  and show that it serves for an  $E_\infty$  version of a generalized  $BP\langle 2 \rangle$ .

The theorem 1.1 involves the elliptic character map. This map originates from the Tate curve

$$y^2 + xy = x^3 + B(q)x + C(q)$$

where

$$B(q) = -1/48(E_4(q) - 1) \text{ and } C(q) = 1/496(E_4(q) - 1) - 1/864(E_6(q) - 1)$$

are integral power series in  $q$ . The evaluation of an ordinary modular form on the Tate curve with its canonical differential corresponds to its  $q$ -expansion. The Tate curve has multiplicative reduction. In order to get a  $\Gamma_1(n)$  structure, that is a point of order  $n$ , one can use the extension of scalars  $\mathbb{Z}((q)) \rightarrow \mathbb{Z}((q)); q \mapsto q^n$ . The resulting curve is usually denoted by  $Tate(q^n)$  and its multiplicative reduction furnishes the Miller character map [Mil89][Lau99]

$$\lambda : TMF_1(n) \longrightarrow K[1/n]((q)).$$

In homotopy this map is the classical  $q$ -expansion. The following result tells us that the  $v_1$ -local character map is faithfully flat in the category of  $BP_*BP$ -comodules:

**Lemma 2.5.** *The map*

$$\lambda : TMF_1(3)[a_1^{-1}]_*X \longrightarrow K[1/3]((q))_*X$$

*is a monomorphism for all spaces  $X$ .*

*Proof.* The  $q$ -expansion of  $v_1 = a_1$  starts with 1 and hence is invertible in the target. Every comodule is the inductive limit of its finitely generated subcomodules. Hence, we may assume that  $BP_*X$  has a finite Landweber filtration  $(F_k)$  with subsequent quotients of the form  $BP_*/I_t$  with  $I_t$  the invariant prime ideal  $(p, v_1, \dots, v_{t-1})$ . By the  $q$ -expansion principle the character map is injective and injective mod  $p$ . Hence it is injective when tensored with each of the quotients  $BP_*/I_t$ . The claim follows from the obvious inductive argument.  $\square$

**Remark 2.6.** It would be interesting to set up character maps for finer structures like  $\Gamma_0(n)$ . We hope to come back to this question in a subsequent work.

### 3. THE $TMF_1(3)$ COHOMOLOGY OF $BSpin$

In this section we will compute the  $TMF_1(3)$  cohomology of the space  $BSpin$ . The main ingredient is the Morava  $K(2)$  homology which has been computed by Kitchloo-Laures in [KL02]. We can use this result to obtain information about the  $E(k, n)$ -cohomology by methods of Ravenel-Wilson-Yagita. A chromatic argument enables us to compute the universal coefficients spectral sequence for  $TMF_1(3)^*BSpin$ . We will show that it collapses at the  $E_2$  term and obtain Theorem 1.1 from the classical  $q$ -expansion principle. Some of the results of this section apply to other situations and hence are formulated more generally than actually needed.

From now on we fix a prime  $p$ . In a first step we do not want to deal with  $\lim^1$  questions and hence work with  $p$ -completed spectra. Let  $BP\langle n \rangle$  be a generalized  $BP\langle n \rangle$  spectrum (in the sense of [LN12]) with coefficient ring  $BP\langle n \rangle^* \cong \mathbb{Z}_p[v_1, \dots, v_n]$ . Let  $I_k$  be the invariant prime ideal  $(p, v_1, \dots, v_{k-1})$  and let  $E(k, n)$  be the spectrum with coefficient ring

$$E(k, n)^* \cong v_n^{-1} BP\langle n \rangle^* / I_k.$$

By definition we have  $E(n, n) \cong K(n)$  and  $E(0, 2) = E(2)$ .

**Remark 3.1.** Recall from Hovey and Sadofsky [HS99a] Theorem 3.4 that for all generalized  $K(n)$  there is a faithfully flat extension of its coefficient ring over which its formal group law becomes strictly isomorphic to the Honda formal group law. This allows us to carry over results from the classical to the generalized  $K(n)$ .

In the following for a fixed  $0 < k \leq n$  let  $X$  be a space with even  $E(k, n)$  cohomology. The exact sequence

$$\begin{aligned} (6) \quad 0 &\longrightarrow E(k-1, n)^{ev} X \xrightarrow{v_{k-1}} E(k-1, n)^{ev} X \longrightarrow E(k, n)^{ev} X \\ &\longrightarrow E(k-1, n)^{odd} X \xrightarrow{v_{k-1}} E(k-1, n)^{odd} X \longrightarrow 0 \end{aligned}$$

tells us that each element in  $E(k-1, n)^{odd} X$  is infinitely divisible by  $v_{k-1}$ . The following result applies since its proof carries over to generalized Johnson-Wilson spectra:

**Theorem 3.2** ([RWY98]4.11). *If  $x$  is infinitely divisible by  $v_{k-1}$  in  $E(k-1, n)^* X$ , then it is zero.*

Our investigation gives:

**Corollary 3.3.** *If  $X$  is a space with even Morava  $K(n)$  cohomology then  $E(k, n)^* X$  is even for all  $k$  and the exact sequences (6) are short exact.*

Next we look at the universal coefficient spectral sequence for  $E = E(n)$

$$Ext_{E_*}(E_*X, E_*) \implies E^*X$$

**Lemma 3.4.** *The global dimension of  $E_* = E(n)_*$  in the category of graded modules equals  $n$ .*

*Proof.* The corresponding result is well known in the ungraded setting. Its graded version is harder to find in the literature but the proof given in [Eis95] 19.5 carries over: Let  $k_*$  be the graded field  $\mathbb{F}_p[v_n^\pm]$ . Since  $(p, v_1, \dots, v_{n-1})$  is a regular sequence the Koszul complex provides a free graded resolution of length  $n$ . This implies the vanishing of  $Tor_{i+1}^{E_*}(k_*, M_*)$  for all  $i \geq n$  and for all  $M_*$ . Next let  $F = (F_n, \varphi_n)$  be a graded minimal free resolution of a finitely generated module  $M_*$ . (The minimality condition means that for each  $n$  a basis of  $F_n$  maps to a minimal set of generators of  $\ker \varphi_{n-1}$ .) Since all differentials in  $k_* \otimes F$  are 0 we have

$$Tor_{i+1}^{E_*}(k_*, M_*) \cong k_* \otimes_{E_*} F_{i+1,*}.$$

This vanishes iff  $F_{i+1,*}$  vanishes because the resolution is free. This shows the claim for finitely generated modules. The general result follows from the graded version of Auslander's theorem [Eis95] Theorem 19.1.  $\square$

**Lemma 3.5.** *Let  $E$  be  $E(2)$  and suppose  $v_1^{-1}E_*X$  is concentrated in even degrees. Then for all even  $t$  we have an isomorphism*

$$Ext_{E_*}^{2,t}(E_*X, E_*) \cong (E/p^\infty)^{t+1}X.$$

*Proof.* The short exact sequence

$$E_* \twoheadrightarrow p^{-1}E_* \twoheadrightarrow E_*/p^\infty$$

gives for all  $s \geq 1$  isomorphisms

$$Ext_{E_*}^{s+1}(E_*X, E_*) \cong Ext_{E_*}^s(E_*X, E_*/p^\infty).$$

By 3.4 these groups vanish for  $s \geq 2$ . Moreover,  $Hom_{E_*}^{t-1}(E_*X, E_*/p^\infty)$  vanishes for even  $t$  because it injects into  $Hom_{E_*}^{t-1}(v_1^{-1}E_*X, v_1^{-1}E_*/p^\infty)$ . Hence the claim follows from the universal coefficient spectral sequence for  $(E/p^\infty)^*X$ .  $\square$

We now specify to the case  $X = BSpin$  and prove a splitting principle.

**Theorem 3.6** ([KL02] 1.2). *For  $n = 1, 2$  let  $b_i \in K(n)_{2i}BSpin$  be the image of the class dual to  $c_1^i$  under the map induced by the inclusion of the maximal torus*

$$BS^1 \longrightarrow BSpin(3) \longrightarrow BSpin.$$

*Then we have*

$$K(n)_*BSpin \cong \pi_*K(n)[b_{2^n \cdot 2}, b_{2^n \cdot 4}, b_{2^n \cdot 6}, \dots].$$

**Corollary 3.7.** *Let  $T(\lfloor n/2 \rfloor)$  be the maximal torus of  $Spin(n)$  and set  $BT^\infty = \text{colim } BT(\lfloor n/2 \rfloor)$ . Then the restriction map from  $K(n)_*BSpin$  to  $K(n)^*BT^\infty$  is injective.*

*Proof.* It suffices to show that the dual map  $K(n)_*BT^\infty \longrightarrow K(n)_*BSpin$  is surjective. This is immediate from the theorem since each monomial  $b_{i_1}b_{i_2} \cdots b_{i_k}$  comes from the classifying space of the  $i_1 + i_2 + \cdots + i_k$ -dimensional torus.  $\square$

**Corollary 3.8.** *For  $E = E(k, n)$  with  $n \leq 2$  the restriction map from  $E^*BSpin$  to  $E^*BT^\infty$  is injective.*

*Proof.* By the previous corollary and by 3.3 we have a map of short exact sequences

$$\begin{array}{ccccc} E(k-1, n)^* BSpin & \xrightarrow{v_{k-1}} & E(k-1, n)^* BSpin & \twoheadrightarrow & E(k, n)^* BSpin \\ \downarrow & & \downarrow & & \downarrow \\ E(k-1, n)^* BT^\infty & \xrightarrow{v_{k-1}} & E(k-1, n)^* BT^\infty & \twoheadrightarrow & E(k, n)^* BT^\infty \end{array}$$

for which the last vertical map can be assumed to be injective. Hence, any element in  $E(k-1, n)^* BSpin$  which restricts trivially to the torus is divisible by  $v_{k-1}$  via an element which again restricts trivially to the torus. Continuing this way, we see that it must be infinitely divisible by  $v_{k-1}$  and thus has to vanish by 3.2.  $\square$

**Theorem 3.9.** *For  $E = E(k)$  with  $k \leq 2$  the universal coefficient isomorphism*

$$E^* BSpin \cong \text{Hom}_{E_*}(E_* BSpin, E_*)$$

*holds.*

*Proof.* We first look at the case  $k = 1$ . Here, the universal coefficient spectral sequence degenerates to the short exact sequence

$$\text{Ext}_{E(1)_*}^*(E(1)_{*-1} BSpin, E(1)_*) \longrightarrow E(1)^* BSpin \longrightarrow \text{Hom}_{E(1)_*}^*(E(1)_* BSpin, E(1)_*) .$$

The previous results gives the injectivity of the second map because it factorizes over the injection into  $E(1)^* BT^\infty \cong \text{Hom}_{E(1)_*}^*(E(1)_* T^\infty, E(1)_*)$ .

For the case  $k = 2$  we like to apply lemma 3.5 and have to show that  $v_1^{-1} E(2)_* X$  is concentrated in even dimensions. By lemma 2.5 (and convention 2.3) it is enough show this for the theory  $F = K((q))$ . Since  $F$  is an algebra spectrum over  $K$ -theory we have the Thom isomorphism  $F_* BSpin \cong F_* MSpin$ . By the Anderson-Brown-Peterson splitting [ABP67] we know that  $MSpin$  splits into a sum of suspension of connective covers of  $ko$  and  $H\mathbb{Z}/2$ . Since the  $K$  homology vanishes on the Eilenberg MacLane part we stay with the sum of even suspensions of  $ko$ . It is known that the  $K$  homology of  $ko$  is concentrated in even dimensions and so is the  $F$  homology of  $BSpin$ .

The even dimensional  $\text{Ext}^2$  term of the universal coefficient spectral sequence has been identified with the odd part of  $(E(2)/p^\infty)^* BSpin$  in lemma 3.5. Hence, for its vanishing it is enough to show the injectivity of the first map in the exact sequence

$$E(2)^* BSpin \longrightarrow (p^{-1} E(2))^* BSpin \longrightarrow (E(2)/p^\infty)^* BSpin .$$

This again follows from Corollary 3.8. We obtain the short exact sequence

$$\text{Ext}_{E(2)_*}^1(E(2)_{*-1} BSpin, E(2)_*) \longrightarrow E(2)^* BSpin \longrightarrow \text{Hom}_{E(2)_*}^*(E(2)_* BSpin, E(2)_*) .$$

Once more by Corollary 3.8 the kernel of the duality map vanishes.  $\square$

**Corollary 3.10.** *For the 2-completed  $TMF_1(3)$  we have the universal coefficient isomorphisms*

$$TMF_1(3)^* BSpin \cong \text{Hom}(TMF_1(3)_* BSpin, TMF_1(3)_*)$$

*Proof.* The result follows from the theorem and 2.2.  $\square$



*Proof of Theorem 1:* We first show the pullback property for the 2-completed theories. For that, observe that there is a pullback on the coefficients

$$\begin{array}{ccc} TMF_1(3)_* & \longrightarrow & K[1/3]_*((q)) \\ \downarrow & & \downarrow \\ (TMF_1(3)_*)_{\mathbf{Q}} & \longrightarrow & (K_*)_{\mathbf{Q}}((q)) \end{array}$$

Hence, when applying the left exact functor  $\mathrm{Hom}_{BP_*}(BP_*BSpin, \_)$  we still have a pullback. By the previous corollary and Theorem 3.9 each corner satisfies the universal coefficient isomorphism and hence we get the desired pullback diagram.

It remains to show the integral result. For odd primes we still have the appropriate pullback diagram by [Lau99] Theorem 1.12. Moreover, for all spectra  $X$  we have an arithmetic pullback by [Bou79]

$$\begin{array}{ccc} X & \longrightarrow & \prod_p L_{H\mathbb{Z}/p}X \\ \downarrow & & \downarrow \\ L_{H\mathbf{Q}}X & \longrightarrow & L_{H\mathbf{Q}}(\prod_p L_{H\mathbb{Z}/p}X) \end{array}$$

For  $X = TMF_1(3)$  this diagram gives a pullback of  $BSpin$  cohomology groups. Moreover, by what we said before we can enlarge the diagram to a pullback diagram of the form

$$\begin{array}{ccc} TMF_1(3)^*BSpin & \longrightarrow & \prod_p \hat{K}_p^*BSpin((q)) \\ \downarrow & & \downarrow \\ TMF_1(3)_{\mathbf{Q}}^*BSpin & \longrightarrow & \prod_p (\hat{K}_p^*)_{\mathbf{Q}}BSpin((q)) \end{array}$$

The lower diagonal map factorizes over  $K_{\mathbf{Q}}^*BSpin((q))$ . Since the pullback of the right hand side along the factorization map has the desired form the result follows.  $\square$

**Corollary 3.11.** *The integral universal coefficient isomorphism*

$$TMF_1(3)^*BSpin \cong \mathrm{Hom}(TMF_1(3)_*BSpin, TMF_1(3)_*)$$

*holds.*

*Proof.* The right hand side satisfies the pullback property in the diagram of the main theorem.  $\square$

#### 4. PONTRYAGIN CLASSES AND THE COHOMOLOGY OF $BString$

In this section we construct explicit generators in the  $TMF_1(3)$  cohomology rings of  $BSpin$  and  $BString$  with the help of Theorem 1.1. We start with a reminder of the  $KO$ -Pontryagin classes.

Recall from [ABP66] Proposition 4.4(b) that the restriction maps from  $KO(BSO(n))$  to the maximal torus  $K(BT(\lfloor n/2 \rfloor))$  are monomorphisms with images the invariants of the Weyl groups. The  $KO$ -Pontryagin classes are defined by the preimage of the series

$$(7) \quad \prod_{i=1}^{\lfloor n/2 \rfloor} (1 + t(x_i \bar{x}_i)) \in K(BT(\lfloor n/2 \rfloor)).$$

Here,  $x_i = 1 - L_i$  are the first Chern classes of the line bundles  $L_i$  over  $BT(\lfloor n/2 \rfloor)$  in  $K$ -theory. It is known that they freely (topologically) generate

$$KO(BSO) \cong KO(BSpin) \cong K(BSpin).$$

*Proof of Theorem 1.2:* The  $K$ -Pontryagin classes freely generate  $K_{Tate}^*(BSpin)$ . We also know that the classical Pontryagin classes in rational singular cohomology freely generate  $H^*(BSpin; TMF_1(3)_{\mathbb{Q}}^*)$ . They are defined in the same way except that in formula (7) the  $x_i$  are the ordinary first Chern classes.

The multiplicative formal group law over  $K_{Tate}$  is strictly isomorphic to the one coming from the *Tate* curve, that is, the  $q$ -expansion of the curve (2). By [Mil89] there is a natural automorphism of  $K_{Tate}$  which exchanges the two orientations. Hence, when we replace the  $x_i$  in the formula for the  $K$ -Pontryagin classes by the first Chern classes with respect to the new orientation we still have free generators. The same argument holds for rational singular homology because here all formal group laws are strictly isomorphic.

The Pontryagin classes with respect to the curve (2) define unique elements in the pullback diagram of Theorem 1.1 and hence free generators of  $TMF_1(3)^* BString$ . Moreover, these classes are determined by their restrictions to the maximal tori by corollary 3.8. Since the induced maps between the tori of  $Spin(2m)$  and  $SO(m)$  is injective in cohomology we are done.  $\square$

Next we consider the cohomology of  $BString$ . The space  $BString$  is defined as the homotopy fibre of the map  $BSpin \rightarrow K(\mathbb{Z}, 4)$  which kills the lowest homotopy group. In particular, we have a sequence of infinite loop spaces

$$(8) \quad K(\mathbb{Z}/2, 2) \longrightarrow K(\mathbb{Z}, 3) \longrightarrow BString \longrightarrow BSpin.$$

**Theorem 4.1** ([KLW04a][KLW04b]). *The sequence (8) induces an exact sequence of Hopf algebras in Morava  $K(2)$  homology at the prime 2*

$$\begin{aligned} K(2)_* &\longrightarrow K(2)_* K(\mathbb{Z}/2, 2) \longrightarrow K(2)_* K(\mathbb{Z}, 3) \longrightarrow K(2)_* BString \\ &\longrightarrow K(2)_* BSpin \longrightarrow K(2)_* \end{aligned}$$

*Algebraically, we have a four term exact sequence of Hopf algebras*

$$K(2)_* \longrightarrow K(2)_* K(\mathbb{Z}, 3) \longrightarrow K(2)_* BString \longrightarrow K(2)_* BSpin \longrightarrow K(2)_*$$

*which splits. In particular, the module  $K(2)_* BString$  is concentrated in even dimensions.*

We will use this information for the computation of the cohomology ring of  $BString$  with respect to

$$E = L_{K(2)} E(2).$$

The coefficients of  $E$  are given by  $E_* = \mathbb{Z}_p[v_1, v_2^{\pm}]_{I_2}^{\wedge}$ .

**Proposition 4.2** ([HS99b] Proposition 2.5). *Suppose  $X$  is a space with even Morava  $K(2)$  homology. Then  $E^* X$  is pro-free in the category of  $L$ -complete modules, that is, it is the completion with respect to  $I_2$  of a free  $E_*$ -module.*

**Proposition 4.3.** *There is an isomorphism of algebras*

$$E^* \llbracket r \rrbracket \cong E^* K(\mathbb{Z}, 3)$$

*with  $r$  in degree 6.*

*Proof.* The  $K(2)$  cohomology of  $K(\mathbb{Z}, 3)$  has been computed by Ravenel and Wilson in [RW80] (see also Su [Su07] for its  $E(1, 2)$  cohomology.) It is topologically free on a generator of degree 6. Clearly, when restricted to a finite subcomplex of  $K(\mathbb{Z}, 3)$  every reduced class becomes nilpotent. Hence, we obtain an algebra map from  $E^*\llbracket r \rrbracket$  to  $E^*K(\mathbb{Z}, 3)$ . The algebra  $E^*K(\mathbb{Z}, 3)$  is concentrated in even degrees with the Morava  $K(2)$  cohomology as its  $I_2$ -reduction. (This can be seen as before with the exact sequence (6).) Thus the result follows from 4.2 and the following version of Hensel's lemma.  $\square$

**Lemma 4.4.** *Let  $R$  be a graded ring with a unique maximal homogeneous ideal  $m$  and let  $P$  and  $Q$  be pro-free  $R$  modules. Then  $f : P \rightarrow Q$  is an isomorphism if and only if it is so modulo  $m$ .*

*Proof.* Tensor the short exact sequences

$$m^k/m^{k+1} \twoheadrightarrow R/m^{k+1} \twoheadrightarrow R/m^k$$

with  $f$  and use the fact that  $m^k/m^{k+1}$  is a (free) module over  $R/m$ .  $\square$

*Proof of Theorem 1.3:* By 4.1 we find a lift  $r$  of the generator of  $K(2)^*K(\mathbb{Z}, 3)$  to  $K(2)^*BString$ . As in 4.3 we can lift  $r$  further to a class in  $E^*BString$  and obtain an algebra map

$$f : E^*BSpin \hat{\otimes} E^*\llbracket r \rrbracket \longrightarrow E^*BString.$$

This map is an isomorphism by 4.1, 4.2 and the previous lemma.  $\square$

**Remark 4.5.** Let  $E$  be the generalized  $E(2)$  or the 2-local  $TMF_1(3)$ . Then there is a pullback square of cohomology rings

$$\begin{array}{ccc} E^*BString & \longrightarrow & (L_{K(2)}E)^*BString \\ \downarrow & & \downarrow \\ (L_{K(1)}E)^*BString & \longrightarrow & (L_{K(1)}L_{K(2)}E)^*BString \end{array}$$

for which we have computed the three corners:  $K(1)$ -locally the map from  $BSpin$  to  $BString$  is an equivalence because the  $K(1)$ -homology of  $K(\mathbb{Z}, 3)$  vanishes. Hence, the lower horizontal map is the inclusion

$$(L_{K(1)}E)^*\llbracket p_1, p_2, \dots \rrbracket \longrightarrow (L_{K(1)}L_{K(2)}E)^*\llbracket p_1, p_2, \dots \rrbracket$$

but for the right vertical map the image of the class  $r$  is unclear. We will address this question somewhere else.

## 5. APPLICATIONS TO REPRESENTATIONS OF LOOP GROUPS

A  $d$ -dimensional spin manifold  $M$  and a representation  $V$  of  $Spin(d)$  give rise to a vector bundle over  $M$  by associating  $V$  to each fibre of the principal  $Spin(d)$ -bundle over  $M$ . Hence it defines an element in the  $K$ -theory ring  $K(M)$ . Its pushforward  $\pi_!$  to the point is the index of the Dirac operator twisted by  $V$ . The construction induces a ring map

$$(9) \quad Rep(Spin(d)) \longrightarrow K(M)$$

which factorizes over the ring  $KBSpin(d)$  by the map which classifies the stable tangent bundle.

In [Bry90] Brylinski conjectured a similar connection for loop space representations and the elliptic cohomology of the 7-connected cover of  $BSpin(d)$  which is usually denoted by  $BString(d)$ . More precisely, he shows that positive energy representation of  $LSpin(d)$  give rise to a cohomology class for the Tate cohomology  $K_{Tate}(M)$  for every *String* manifold. Its push forward to a point is the formal index of the  $S^1$  equivariant Dirac operator on the loop space of  $M$ . This index is known to be a modular form of some level which leads to the hope that this class refines to a class in topological modular forms.

Let us formulate the conjecture of Brylinski more precisely. Let  $m$  be an integer and let  $P_m(d)$  be the free abelian group generated by the isomorphism classes of irreducible positive energy representations of  $\tilde{LSpin}(d)$  of level  $m$ .

**Conjecture 5.1.** There is an integer  $n$  depending on  $d$  and  $m$  and an additive map

$$\varphi_d : P_m(d) \longrightarrow TMF(n)^0 BString(d)$$

whose Miller character coincides with the bundle obtained by associating the representation to the principal  $\tilde{LSpin}(d)$  bundle over  $BString(n)$ :

$$(10) \quad \lambda\varphi_d(V) = \widetilde{LESpin} \times_{\tilde{LSpin}} V.$$

The evaluation of this characteristic class on the fundamental class of a  $d$  dimensional *String* manifold is the “index of the Dirac operator on  $LM$  with coefficient in the bundle” associated to the representation. Using a result of Kac and Wakimoto he shows

**Theorem 5.2.** [Bry90] *The conjecture 5.1 holds rationally. The integer  $n = 24(m+g)$ , with  $g = d - 2$  the Coxeter number, is high enough.*

We will show a stable version of the integral conjecture. Note that stably the representation rings of *Spin* and *SO* coincide. Moreover, the map (9) factorizes over  $KBSO$ . Only the description in terms of index theory fails for non *Spin* manifolds since  $K$ -theory is not *SO* orientable.

Similar things happen in the loop group situation: the power series  $\lambda\varphi_d(V)$  in (10) already lies in  $R(Spin(d))[[q]]$  (compare [Bry90] 465ff) and its character is invariant under the action of  $\Gamma(n)$  by [KW88] Theorem A.

**Theorem 5.3.** *Let  $P_m$  be the inverse limit of all  $P_m(n)$ . For  $n$  divisible by 3 let  $(P_m)_{\Gamma(n)}$  be the subgroup of  $P_m$  consisting of representations  $V$  with character in modular functions with respect to  $\Gamma(n)$ . Then there is an additive map*

$$\varphi : (P_m)_{\Gamma(n)} \longrightarrow TMF(n)^* BSpin$$

*whose Miller character yields the desired class. Moreover, for all String manifolds  $M$  the class  $\pi_!(\varphi(V)TM)$  is the formal index of the  $S^1$  equivariant Dirac operator on the loop space of  $M$  twisted by  $V$ .*

We prepare the proof with two lemmas.

**Lemma 5.4.** *Let 3 be inverted in the following rings of modular forms and let  $n$  be divisible by 3. Then the following ring extensions are flat and finite:*

$$(11) \quad M_{\Gamma_1(3)} \longrightarrow M_{\Gamma(3)}$$

$$(12) \quad M_{\Gamma(3)} \longrightarrow M_{\Gamma(n)}$$

*Proof.* This follows from [KM85]5.5.1 since the moduli problem for  $\Gamma_1(3)$  structures is representable.  $\square$

**Lemma 5.5.** *Let  $R$  be coherent,  $M$  be a finitely generated  $R$  module and  $N$  be a flat  $R$ -module. Then we have the isomorphism*

$$\mathrm{Hom}_R(M, R) \otimes_R N \cong \mathrm{Hom}_R(M, N).$$

*Proof.* Choose a finitely generated free presentation  $F_*$  of  $M$ . Since  $N$  is flat the left hand side of the claim is the kernel of

$$\mathrm{Hom}_R(F_0, R) \otimes_R N \longrightarrow \mathrm{Hom}_R(F_1, R) \otimes_R N.$$

By finiteness it coincides with the kernel of

$$\mathrm{Hom}_R(F_0, N) \longrightarrow \mathrm{Hom}_R(F_1, N).$$

which is the right hand side.  $\square$

*Proof of 5.3:* By what we said before it is enough to show that the diagram

$$\begin{array}{ccc} TMF(n)^* BSpin & \xrightarrow{\lambda} & K_{Tate}^* BSpin \\ \downarrow & & \downarrow \\ H^*(BSpin, TMF(n)^*_{\mathbf{Q}}) & \longrightarrow & H^*(BSpin, K_{Tate}^*_{\mathbf{Q}}) \end{array}$$

is a pullback for all  $n$  divisible by 3. This follows as before from the universal coefficient isomorphism:

$$(13) \quad TMF(n)^* BSpin \cong TMF_1(3)^* BSpin \hat{\otimes}_{M_{\Gamma_1(3)}} M_{\Gamma(n)}$$

$$(14) \quad \cong \mathrm{Hom}(TMF_1(3)^* BSpin, M_{\Gamma_1(3)}) \hat{\otimes}_{M_{\Gamma_1(3)}} M_{\Gamma(n)}$$

$$(15) \quad \cong \mathrm{Hom}(TMF_1(3)^* BSpin, M_{\Gamma(n)}).$$

Here, the first isomorphism holds for all finite spectra by the proposition 5.4. It also holds for  $BSpin$  since there isn't any  $\lim^1$ . The second isomorphism follows from 3.11 and the last one again is consequence of 5.4 and the previous lemma after passing to the limit of finite subspectra of  $BSpin$ .  $\square$

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